Gaussian Wigner distributions and hierarchies of nonclassical states in quantum optics-The single mode case

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A recently introduced hierarchy of states of a single mode quantised radiation field is examined for the case of centered Guassian Wigner distributions. It is found that the onset of squeezing among such states signals the transition to the strongly nonclassical regime. Interesting consequences for the photon number distribution, and explicit representations for them, are presented.

I. INTRODUCTION

Squeezed states of light, and other states exhibiting either antibunching or subpoissonian photon statistics or both, are well known examples of so called "nonclassical" states of radiation [1] [2] [3] [4]. In fact these are the simplest and most familiar ones out of an infinite hierarchy of independent signatures of nonclassical states in quantum optics; many other signatures have been presented in the literature [5].

The precise definition of a nonclassical state of radiation is based upon the diagonal coherent state expansion of the density matrix $\hat{\rho}$ of the state in the quantum theory. Limiting ourselves to the single mode radiation field this expansion is [6]

$$\hat{\rho} = \int \frac{d^2z}{\pi}\phi(z)|z\rangle\langle z| \, , \tag{1.1}$$

where the coherent states $|z\rangle$ are the familiar normalised eigenstates of the photon annihilation operator $\hat{a}$ with complex eigenvalue z and $\phi(z)$ is a real normalised weight function which is in general a distribution. The state $\hat{\rho}$ is said to be "classical " if $\phi(z)$ is pointwise nonnegative, and nowhere more singular than a delta function, so that it can be interpreted as a classical probability density over the complex plane. Otherwise $\hat{\rho}$ is a "nonclassical " state. This classification is clearly invariant under rotations and translations in phase space.

It has been shown elsewhere that there is a dual operator based approach to this distinction between classical and nonclassical states, which is physically quite instructive [7]. The representation (1.1), as is well known, is closely related to the normal ordering rule of correspondence between classical dynamical variables and quantum operators. Given any real classical function $f(z^\star, z)$ of a complex variable z and its conjugate, one defines a hermitian operator $\hat{F}$ in quantum theory by the replacement $z \to \hat{a}, z^\star \to \hat{a}^\dagger$ and then bringing all factors $\hat{a}^\dagger$ "by hand" to the left of all factors $\hat{a}$:

$$\begin{aligned} f(z^\star, z) \to \hat{F} &= f(\hat{a}^\dagger, \hat{a})|_{\hat{a}^\dagger \text{ to left, } \hat{a} \text{ to right}}, \\ \langle z|\hat{F}|z\rangle &= f(z^\star, z). \end{aligned} \tag{1.2}$$

Then the quantum mechanical expectation value of $\hat{F}$ in the state $\hat{\rho}$ is

$$\langle\hat{F}\rangle = \mathrm{Tr}(\hat{\rho}\hat{F}) = \int \frac{d^2z}{\pi}\phi(z)f(z^\star, z). \tag{1.3}$$

The key observation now is that while the correspondence $f \leftrightarrow \hat{F}$ is linear and takes real functions to hermitian operators and vice versa, a real nonnegative $f(z^\star, z)$ may well lead to a hermitian indefinite $\hat{F}$. A state $\hat{\rho}$ is then said to be classical if this permitted " quantum negativity " in operators never shows up in expectation values, nonclassical otherwise:

$$\begin{aligned} \hat{\rho} \quad \text{Classical} &\Leftrightarrow \mathrm{Tr}(\hat{\rho}\hat{F}) \geq 0 \ \text{ for every } f(z^\star, z) \geq 0, \\ \hat{\rho} \quad \text{Nonclassical} &\Leftrightarrow \mathrm{Tr}(\hat{\rho}\hat{F}) < 0 \ \text{ for some } f(z^\star, z) \geq 0. \end{aligned} \tag{1.4}$$

With this alternative characterization (completely equivalent to the usual one), one has the possibility of defining several degrees or levels of nonclassicality, if one restricts in various ways the collection of operators $\hat{F}$ for which one tests the conditions given in (1.4) [7]. Specifically, for a single mode system, it has been shown by considering the subset of phase invariant (number conserving) operators $\hat{F}$ which arise from $f(z^\star, z)$ obeying

$$f(z^\star e^{-i\alpha}, z e^{i\alpha}) = f(z^\star, z) \tag{1.5}$$

that an exhaustive and mutually exclusive three-fold classification of states is possible. If $f(z^\star, z)$ obeys (1.5), then for the expectation value of the corresponding $\hat{F}$ it suffices to use an angle average of $\phi(z)$:

$$\begin{aligned} [\hat{F}, \hat{a}^\dagger \hat{a}] = 0 \Rightarrow \mathrm{Tr}(\hat{\rho}\hat{F}) &= \int_0^\infty dI \mathcal{P}(I)\, f(I^{1/2}, I^{1/2}), \\ \mathcal{P}(I) &= \int_0^{2\pi} \frac{d\theta}{2\pi} \phi(I^{1/2} e^{i\theta}). \end{aligned} \tag{1.6}$$

One can then obtain the following finer classification of all states:

$$\begin{aligned} \hat{\rho} \quad \text{Classical} &\Leftrightarrow \phi(z) \geq 0, \ \text{ so } \mathcal{P}(I) \geq 0, \\ \hat{\rho} \quad \text{Weakly nonclassical} &\Leftrightarrow \mathcal{P}(I) \geq 0, \ \text{ but } \phi(z) \ngeq 0 \\ \hat{\rho} \quad \text{Strongly nonclassical} &\Leftrightarrow \mathcal{P}(I) \ngeq 0, \ \text{ so } \phi(z) \ngeq 0 \end{aligned} \tag{1.7}$$

Thus the previous "nonclassical" has been subdivided now into "weakly nonclassical" and "strongly nonclassical" states. Upto and including the weakly nonclassical level, $\mathcal{P}(I)$ can be treated as a classical probability density for intensity, whether or not $\phi(z)$ can be regarded as a probability distribution over the complex plane; in the third strongly nonclassical regime, even $\mathcal{P}(I)$ ceases to be a probability density.

The aim of this paper is to illustrate these ideas in the concrete case of states described by Gaussian-Wigner distributions on phase space. It is well known that in a wide variety of physical processes the states of radiation that are produced are indeed of this type [8]. Their description also lends itself to direct analytical treatment. What we shall demonstrate is that within this set of states, the onset of squeezing signals an abrupt change from classical to the strongly nonclassical regime; thus the weakly nonclassical states do not show up at all in this family!

The material of this paper is arranged as follows. In Section II we trace the connection between the descriptions of an operator via its Weyl weight and its Wigner representative, and the diagonal weight $\phi(z)$. This gives us a clear picture of the extent to which $\phi(z)$ can be a singular distribution, and in turn how singular the quantity $\mathcal{P}(I)$ can in principle be. Section III examines the class of centered Gaussian Wigner distributions. These are fully parametrised by the variance or noise matrix which has to be positive semidefinite and also must obey the uncertainty principle. Among these states the only two qualitatively different ones are the nonsqueezed and squeezed ones. In the former case, both $\phi(z)$ and $\mathcal{P}(I)$ can be computed explicitly, and as expected they are finite nonnegative normalized functions. This is consistent with their being classified as classical states. In contrast, the squeezed states are shown to be strongly nonclassical, and one never sees the weakly nonclassical possibility at all. Section IV gives an example of weakly nonclassical states which are naturally outside the Gaussian Wigner family, and offers some concluding remarks.

II. NATURE OF THE DISTRIBUTIONS $\phi(z)$ AND $\mathcal{P}(I)$

It is useful to begin by recalling the general properties of the diagonal weight $\phi(z)$ and its angular average $\mathcal{P}(I)$, and by giving an indication of the kinds of singular distributions we must be prepared to encounter [9]. This is best

done by viewing the set of all possible density matrices $\hat{\rho}$ as a subset of the family of Hilbert-Schmidt (H-S) operators. An operator A on Hilbert space is of H-S type if

$$\mathrm{Tr}(A^\dagger A) < \infty, \tag{2.1}$$

and among H-S operators we have a natural inner product :

$$(A, B) = \mathrm{Tr}(A^\dagger B). \tag{2.2}$$

We deal throughout with systems of one degree of freedom, and with the annihilation and creation operators $\hat{a}, \hat{a}^\dagger$ related to hermitian $\hat{q}$ and $\hat{p}$ in the standard way:

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p}) \tag{2.3}$$

The unitary phase space displacement operators are defined by and have the following properties:

$$\begin{aligned} D(\sigma,\tau) &= \exp(i\sigma\hat{q} - i\tau\hat{p}), \quad -\infty < \sigma, \tau < \infty; \\ D(\sigma,\tau)^\dagger &= D(\sigma,\tau)^{-1} = D(-\sigma,-\tau); \\ \mathrm{Tr}\left(D(\sigma',\tau')^\dagger D(\sigma,\tau)\right) &= 2\pi\delta(\sigma'-\sigma)\delta(\tau'-\tau). \end{aligned} \tag{2.4}$$

Any H-S operator A can be expanded in the form of an operator Fourier integral representation using its "Weyl weight" $\tilde{A}(\sigma,\tau)$ as expansion coefficient [10]:

$$\begin{aligned} A &= \int\int \frac{d\sigma d\tau}{\sqrt{2\pi}} \tilde{A}(\sigma,\tau) D(\sigma,\tau), \\ \tilde{A}(\sigma,\tau) &= \frac{1}{\sqrt{2\pi}} \left(D(\sigma,\tau), A\right), \\ \mathrm{Tr}(A^\dagger A) = (A, A) &= \int\int d\sigma\ d\tau |\tilde{A}(\sigma,\tau)|^2. \end{aligned} \tag{2.5}$$

Thus the H-S property (2.1) of A is translated exactly into the L^2 property of $\tilde{A}(\sigma,\tau)$ over $\mathcal{R}^2$.

From $\tilde{A}(\sigma,\tau)$ we pass to the Wigner representative or Wigner distribution $W(q,p)$ of the operator A by a double Fourier transform at the c-number level [11]:

$$W(q,p) = \int\int \frac{d\sigma d\tau}{(2\pi)^{\frac{3}{2}}} \tilde{A}(\sigma,\tau) \exp\left(i\sigma q - i\tau p\right) \tag{2.6}$$

Here q and p are canonical coordinates over a classical phase space, and in case A is hermitian its Wigner representative $W(q,p)$ is real. Now the H-S property for A amounts to $W(q,p)$ being an L^2 function over $\mathcal{R}^2$:

$$Tr(A^\dagger A) = (A, A) = 2\pi \int\int dqdp |W(q,p)|^2 \tag{2.7}$$

For density matrices we are also interested in the ordinary trace:

$$Tr(A) = \sqrt{2\pi}\tilde{A}(0,0) = \int\int dqdp W(q,p) \tag{2.8}$$

It is in the passage from $\tilde{A}(\sigma,\tau)$ or $W(q,p)$ to $\phi(z)$ that the distribution character of the latter shows up. From the diagonal representation

$$A = \int \frac{dxdy}{2\pi}\ \phi(z)\ |z\rangle\langle z|, \tag{2.9}$$

where $z = \frac{1}{\sqrt{2}}(x + iy)$, when we connect up with the previous relations (2.5, 2.6) we get the result:

$$\begin{aligned} \phi(z) &= \int\int \frac{d\sigma d\tau}{\sqrt{2\pi}} e^{\frac{1}{4}(\sigma^2+\tau^2)} \tilde{A}(\sigma,\tau) e^{i(\sigma x - \tau y)} \\ &= \int\int \frac{d\sigma d\tau}{2\pi} e^{\frac{1}{4}(\sigma^2+\tau^2) + i(\sigma x - \tau y)} \int\int dqdp W(q,p) e^{i(\tau p - \sigma q)}. \end{aligned} \tag{2.10}$$

Thus the most singular kind of $\phi(z)$ is one whose Fourier transform is the increasing Gaussian factor $\exp \frac{1}{4}(\sigma^2+\tau^2)$ times a square integrable function $\tilde{A}(\sigma,\tau)$ - this is the worst behaviour that can in principle occur. Conversely for a classical state $\tilde{A}(\sigma,\tau)$ must more than overwhelm this exponential factor and moreover yield a nonnegative $\phi(z)$.

Let us next see what this situation for $\phi(z)$ entails for its angular average $\mathcal{P}(I)$. We work directly with the Wigner distribution $W(q,p)$ and find after performing the angular integration:

$$\begin{aligned}\mathcal{P}(I) &= \int_0^{2\pi} \frac{d\theta}{2\pi}\phi(I^{\frac{1}{2}}e^{i\theta}) \\ &= \int\int \frac{d\sigma d\tau}{2\pi} e^{\frac{1}{4}(\sigma^2+\tau^2)} J_0(\sqrt{2I(\sigma^2+\tau^2)}) \int\int dqdpW(q,p)e^{i(\tau p-\sigma q)}. \end{aligned} \tag{2.11}$$

If we substitute $\sigma = \sqrt{2K}\cos\psi$, $\tau = \sqrt{2K}\sin\psi$, we can carry out one more angular integration and bring $\mathcal{P}(I)$ to the following form:

$$\begin{aligned}\mathcal{P}(I) &= \int_0^\infty dK e^{\frac{K}{2}}.J_0(2\sqrt{IK}) \int\int dqdpW(q,p)\ J_0(\sqrt{2K(q^2+p^2})) \\ &= \int_0^\infty dK e^{\frac{K}{2}}.J_0(2\sqrt{IK}) \int_0^\infty dL J_0(2\sqrt{KL}) \int_0^{2\pi} d\chi W(\sqrt{2L}\cos\chi, \sqrt{2L}\sin\chi) \end{aligned} \tag{2.12}$$

Now just as the relation (2.10) between $\phi(z)$ and $\tilde{A}(\sigma,\tau)$ involved the classical two dimensional Fourier transformation, here one is concerned with the single variable Fourier-Bessel transformation over the half-line $(0,\infty)$ which states [12]:

$$\begin{aligned} &\int_0^\infty dI|f(I)|^2 < \infty \Rightarrow \\ f(I) &= \int_0^\infty dK g(K) J_0(2\sqrt{IK}), \\ g(K) &= \int_0^\infty dI f(I) J_0(2\sqrt{IK}), \\ \int_0^\infty dI|f(I)|^2 &= \int_0^\infty dK|g(K)|^2, \\ \int_0^\infty dK J_0(2\sqrt{LK})J_0(2\sqrt{IK}) &= \delta(I-L). \end{aligned} \tag{2.13}$$

This means that the most singular possible behaviour for $\mathcal{P}(I)$ which can in principle occur is that its Fourier-Bessel transform can be the factor $e^{\frac{K}{2}}$ times a square integrable function of K over the domain $(0,\infty)$, namely the Fourier-Bessel transform of the angular average of $W(q,p)$. The factor $e^{\frac{K}{2}}$ is just the earlier factor $e^{\frac{1}{4}(\sigma^2+\tau^2)}$ present in eq (2.10); and the situation for $\mathcal{P}(I)$ is marginally better than for $\phi(z)$ since now only the angular average of $\phi(z)$ is involved.

The use of phase space language in describing operators in quantum mechanics leads naturally to an examination of the behaviours of $\phi(z)$ and $\mathcal{P}(I)$ under phase space rotations and translations. As is easy to see, their behaviour under rotations is simple:

$$W'(q,p) = W(q\cos\alpha - p\sin\alpha, p\cos\alpha + q\sin\alpha) \Leftrightarrow \phi'(z) = \phi(ze^{i\alpha}) \Rightarrow \mathcal{P}'(I) = \mathcal{P}(I). \tag{2.14}$$

This invariance of $\mathcal{P}(I)$ is as expected. Under translations we have

$$W'(q,p) = W(q-q_0, p-p_0) \Leftrightarrow \phi'(z) = \phi(z-z_0), \quad z_0 = \frac{1}{\sqrt{2}}(q_0+ip_0) \tag{2.15}$$

However now $\mathcal{P}'(I)$ is not expressible in terms of $\mathcal{P}(I)$ alone as phase sensitivity is introduced by a translation. Therefore while our threefold classification scheme (1.7) is obviously invariant under phase space rotations, the behaviour with respect to translations is much more subtle.

It is evident that the classical states with both $\phi(z)$ and $\mathcal{P}(I)$ nonnegative remain classical under translations. However a weakly nonclassical state becomes strongly nonclassical for a suitably chosen translation, as the following physical argument shows. At the origin $\mathcal{P}(0)$ reduces to $\phi(0)$ as no angular average remains. If a weakly nonclassical state is given, its $\phi(z)$ must become effectively negative somewhere in the complex plane. By translating the origin to such a point and then computing $\mathcal{P}'(0)$ we see that the resulting state is strongly nonclassical. Following a similar argument we also see that we can recover $\phi(z)$ in its entirety by subjecting the initial state to all possible phase space displacements z_0, $\phi'(z) = \phi(z - z_0)$, and then computing the resulting $\mathcal{P}'(I)$ and collecting the results.

We conclude this Section by relating the distribution $\mathcal{P}(I)$ to the photon number probabilities. Indeed these involve a complete independent set of phase insensitive quantities and their expectation values:

$$\begin{aligned} f(z^\star, z) &= e^{-z^\star z}\,\frac{(z^\star z)^n}{n!} \leftrightarrow \hat{F} = |n\rangle\langle n|, \\ p(n) &= \mathrm{Tr}(\hat{\rho}\hat{F}) = \langle n|\hat{\rho}|n\rangle \\ &= \int_0^\infty dI \mathcal{P}(I) e^{-I}\,\frac{I^n}{n!} \end{aligned} \tag{2.16}$$

These $p(n)$'s always give well defined normalised probabilities for finding various numbers of photons, whether or not $\mathcal{P}(I)$ is itself a probability density. Formally one can invert the above to get $\mathcal{P}(I)$ in terms of $p(n)$, as indeed one would expect. If we define the generating function $q(K)$ by

$$q(K) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} K^n p(n) \tag{2.17}$$

we see that $q(K)$ converges for all real K and is related to $\mathcal{P}(I)$ by

$$\begin{aligned} q(K) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} K^n \int_0^\infty dI \mathcal{P}(I) e^{-I}\,\frac{I^n}{n!} \\ &= \int_0^\infty dI \mathcal{P}(I) e^{-I} J_0(2\sqrt{IK}). \end{aligned} \tag{2.18}$$

Using the formula (2.13) of the Fourier Bessel transformation again we get the inversion

$$\mathcal{P}(I) = e^I \int_0^\infty dK q(K) J_0(2\sqrt{IK}). \tag{2.19}$$

In the classical and weakly nonclassical cases, then, the generating function $q(K)$ is itself well behaved and leads to nonnegative $\mathcal{P}(I)$, but in the strongly nonclassical case, it causes $\mathcal{P}(I)$ to be a distribution, or at any rate not a probability.

III. THE CASE OF GAUSSIAN WIGNER DISTRIBUTIONS

We consider the family of centered Gaussian Wigner distributions, namely those which have vanishing means for q and p [13]. The most general such distribution is determined by a real symmetric 2×2 matrix G

$$\begin{aligned} W_G(q,p) &= \frac{\sqrt{\mathrm{det}G}}{\pi} \exp\left(-(q \quad p)\, G \begin{pmatrix} q \\ p \end{pmatrix} \right), \\ G &= \begin{pmatrix} A & B \\ B & C \end{pmatrix}. \end{aligned} \tag{3.1}$$

The condition that $W_G(q,p)$ represent a physically realisable quantum mechanical state imposes the following restrictions on G corresponding respectively to normalisability and the uncertainty principle [14]:

$$G > 0, \quad \text{ie} \quad A + C > 0, \quad \Delta = \det G = AC - B^2 > 0; \tag{3.2a}$$

$$G^{-1} + i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \geq 0 \ , \ \text{ie} \quad A + C \geq 0 \ , \quad \Delta \geq \Delta^2. \tag{3.2b}$$

Combining these we have the complete set of restrictions on G given by

$$A + C > 0, \quad 0 < \Delta \leq 1. \tag{3.3}$$

The noise or variance matrix V is defined and given by

$$\begin{aligned} V &= \begin{pmatrix} (\Delta q)^2 & \Delta(q,p) \\ \Delta(q,p) & (\Delta p)^2 \end{pmatrix} = \frac{1}{2} G^{-1} = \frac{1}{2\Delta} \begin{pmatrix} C & -B \\ -B & A \end{pmatrix} \\ (\Delta q)^2 &= \int\int dq \ dp \ q^2 \ W_G(q,p), \\ \Delta(q,p) &= \int\int dq \ dp \ qp \ W_G(q,p), \\ (\Delta p)^2 &= \int\int dq \ dp \ p^2 \ W_G(q,p). \end{aligned} \tag{3.4}$$

Here the vanishing of the means of q and p has been used. In terms of V, the uncertainty principle appears in the following form [15]:

$$\det V = \frac{1}{4\Delta} \geq \frac{1}{4}. \tag{3.5}$$

We can use the covariance of $\phi(z)$ and the invariance of $\mathcal{P}(I)$ under phase space rotations to simplify the situation and to assume without loss of generality that G and V are diagonal. Moreover these rotations do not disturb the three-fold classification of states (1.7). Therefore we parametrise G and V using two real positive parameters α and β as follows:

$$\begin{aligned} V &= \frac{1}{2} \begin{pmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{pmatrix} \\ G &= \begin{pmatrix} 1/\alpha^2 & 0 \\ 0 & 1/\beta^2 \end{pmatrix}, \ \alpha, \beta > 0, \alpha\beta \geq 1; \\ W_{(\alpha,\beta)}(q,p) &= \frac{1}{\pi\alpha\beta} \exp\left(-\frac{q^2}{\alpha^2} - \frac{p^2}{\beta^2} \right). \end{aligned} \tag{3.6}$$

To deal with $\phi(z)$ and $\mathcal{P}(I)$ we need respectively the Fourier transform and the angular average of $W_{(\alpha,\beta)}(q,p)$; these are:

$$\int\int dq dp W_{(\alpha,\beta)}(q,p) exp(i\tau p - i\sigma q) = \exp\left(-\frac{\alpha^2\sigma^2}{4} - \frac{\beta^2\tau^2}{4} \right); \tag{3.7a}$$

$$\int_0^{2\pi} d\chi W_{(\alpha,\beta)}(\sqrt{2L}\cos\chi, \sqrt{2L}\sin\chi) = \frac{2}{\alpha\beta} \exp\left(-L(\frac{1}{\alpha^2} + \frac{1}{\beta^2}) \right) I_0 \left(L(\frac{1}{\alpha^2} - \frac{1}{\beta^2}) \right) \tag{3.7b}$$

Here $I_0(w) = J_0(iw)$ is the Bessel function of order zero and imaginary argument.

Returning to the Wigner function $W_{(\alpha,\beta)}(q,p)$, the nonsqueezed case corresponds to both $\alpha, \beta \geq 1$; while if one of them becomes less than unity we have a squeezed state. For definiteness in the latter case we take p to be the squeezed variable, so we take $\beta < 1$ and $\alpha > 1$ maintaining $\alpha\beta \geq 1$. Formally we have throughout, on combining eqs (2.10 , 3.7a):

$$\phi_{(\alpha,\beta)}(z) = \int\int \frac{d\sigma d\tau}{2\pi} e^{i(\sigma x - \tau y)} \exp\left[-\frac{1}{4}(\alpha^2 - 1)\sigma^2 - \frac{1}{4}(\beta^2 - 1)\tau^2 \right] \tag{3.8}$$

In the nonsqueezed regime these integrals can be computed and we get expected results:

$$\phi_{(\alpha,\beta)}(z) = \begin{cases} 2(\alpha^2-1)^{-1/2}.(\beta^2-1)^{-1/2}\exp\left[-\dfrac{x^2}{\alpha^2-1}-\dfrac{y^2}{\beta^2-1}\right], & \alpha,\beta>1; \\ \sqrt{2\pi}\delta(x)\sqrt{2}(\beta^2-1)^{-1/2}\exp\left(-\dfrac{y^2}{\beta^2-1}\right), & \alpha=1,\beta>1; \\ \sqrt{2\pi}\delta(y)\sqrt{2}(\alpha^2-1)^{-1/2}\exp\left(-\dfrac{x^2}{\alpha^2-1}\right), & \alpha>1,\beta=1; \\ 2\pi\delta(x)\delta(y), \qquad \alpha=\beta=1. \end{cases} \tag{3.9}$$

In all these cases the state is classical. However, once β dips below unity, we see from eq (3.8) that the Fourier transform of $\phi(z)$ is an increasing Gaussian in the variable τ. This means that $\phi(z)$ has switched abruptly to being a distribution, essentially of the most singular kind that can arise. (Of course, if β continually decreases and squeezing increases, $\phi(z)$ does become more and more singular). This is consistent with squeezed states being nonclassical. The interesting point is that there is no intermediate regime (among "Gaussian-Wigner" states) in which the singularity of $\phi(z)$ is somewhat milder, say involving finite order derivatives of delta functions.

To follow the behaviour of $\mathcal{P}_{(\alpha,\beta)}(I)$ as we pass from the nonsqueezed state to the squeezed regime, and when $\beta<1$ to discriminate between the weakly nonclassical and the strongly nonclassical possibilities, we begin by combining eqs (2.12 3.7b) to get a formal integral expression for $\mathcal{P}_{(\alpha,\beta)}(I)$:

$$\mathcal{P}_{(\alpha,\beta)}(I) = \frac{2}{\alpha\beta}\int_0^\infty dK e^{K/2}.J_0(2\sqrt{IK})\int_0^\infty dL\ e^{-L(\frac{1}{\alpha^2}+\frac{1}{\beta^2})}J_0(2\sqrt{LK})I_0\left(L(\frac{1}{\alpha^2}-\frac{1}{\beta^2})\right) \tag{3.10}$$

The first integral, over L, always converges thanks to the asymptotic behaviours of $J_0(z)$ and $I_0(z)$:

$$\begin{aligned} J_0(z) &\underset{z\to+\infty}{\longrightarrow} \sqrt{\frac{2}{\pi z}}\cos(z-\pi/4), \\ I_0(z) &\underset{z\to+\infty}{\longrightarrow} \frac{e^z}{\sqrt{2\pi z}} \end{aligned} \tag{3.11}$$

Moreover, by suitable and permitted analytic continuation of a standard definite integral available in the literature [reference 16 p.711, formula 6.644] we obtain a formula with whose help the L-integral can be done explicitly. The requisite formula is, for real parameters a,b,c obeying $a>|c|\geq 0, b>0$:

$$\int_0^\infty dx\ e^{-ax}\ J_0(2\sqrt{bx})\ I_0(cx) = \frac{1}{\sqrt{a^2-c^2}}\exp\left(\frac{-ab}{a^2-c^2}\right)\ I_0\left(\frac{cb}{a^2-c^2}\right) \tag{3.12}$$

Taking $a=\frac{1}{\alpha^2}+\frac{1}{\beta^2}, b=K, c=\frac{1}{\alpha^2}-\frac{1}{\beta^2}$ here and using the result in eq (3.10) we get for $\mathcal{P}_{(\alpha,\beta)}(I)$ the single integral

$$\mathcal{P}_{(\alpha,\beta)}(I) = \int_0^\infty dK\ e^{K/2}.\ J_0(2\sqrt{IK})\ e^{-K(\frac{\alpha^2+\beta^2}{4})}\ I_0\left(\frac{K}{4}(\alpha^2-\beta^2)\right) \tag{3.13}$$

First let us look at the classical nonsqueezed situation. Leaving aside the marginal cases when α or β equals unity, we again use the result (3.12) to evaluate (3.13) explicitly:

$$\alpha,\beta>1 \quad : \quad \mathcal{P}_{(\alpha,\beta)}(I) = 2(\alpha^2-1)^{-1/2}.(\beta^2-1)^{-1/2}\exp\left[-I\left(\frac{1}{\alpha^2-1}+\frac{1}{\beta^2-1}\right)\right]\ I_0\ \left[I\left(\frac{1}{\alpha^2-1}-\frac{1}{\beta^2-1}\right)\right] \tag{3.14}$$

This is explicitly nonnegative, and is consistent with the state being classical. In this case, we can go further and obtain a closed form expression for the photon-number probabilities $p_{(\alpha,\beta)}(n)$. We have:

$$p_{(\alpha,\beta)}(n) = \int_0^\infty dI\mathcal{P}_{(\alpha,\beta)}(I)e^{-I}\ \frac{I^n}{n!}$$

$$
\begin{aligned}
&= \frac{1}{n!}\,\frac{2}{\sqrt{(\alpha^2-1)(\beta^2-1)}}\int_0^\infty dI e^{-aI} I^n\, I_0(bI),\\
a &= 1+\frac{1}{\alpha^2-1}+\frac{1}{\beta^2-1}=\frac{\alpha^2\beta^2-1}{(\alpha^2-1)(\beta^2-1)},\\
b &= \frac{(\beta^2-\alpha^2)}{(\alpha^2-1)(\beta^2-1)}
\end{aligned}
\tag{3.15}
$$

The resulting integral is a known one leading to an expression in terms of the hypergeometric function [reference 16, p.711, formula 6.621]

$$
\int_0^\infty dx\, e^{-ax}\, x^n\, I_0(bx) = \frac{n!}{a^{n+1}} F\left(\frac{n}{2}+\frac{1}{2}, \frac{n}{2}+1; 1; \frac{b^2}{a^2}\right) \tag{3.16}
$$

so the probabilities $p_{(\alpha,\beta)}(n)$ are:

$$
\begin{aligned}
p_{(\alpha,\beta)}(n) &= \frac{2}{\sqrt{(\alpha^2-1)(\beta^2-1)}}.\left[\frac{(\alpha^2-1)(\beta^2-1)}{\alpha^2\beta^2-1}\right]^{n+1} F\left(\frac{n}{2}+\frac{1}{2}, \frac{n}{2}+1; 1; z\right),\\
z &= \left(\frac{\alpha^2-\beta^2}{\alpha^2\beta^2-1}\right)^2 \quad , \quad \alpha, \beta > 1.
\end{aligned}
\tag{3.17}
$$

The combination z of α and β does not exceed unity as we have $\alpha,\ \beta > 1$:

$$
1 - z = (\alpha^4-1)(\beta^4-1)/(\alpha^2\beta^2-1)^2. \tag{3.18}
$$

It is interesting to note that the result (3.17) for $p_{(\alpha,\beta)}(n)$ is a manifestly nonnegative closed-form expression; in this respect it may be contrasted with the expression given earlier in the literature [8]

Next let us consider the squeezed regime $\beta < 1, \alpha \geq 1/\beta$. Then the exponential factor $e^{K/2}$ in the integral in eq (3.13) overpowers the remaining factors:

$$
e^{K/2}\, e^{-K(\alpha^2+\beta^2)/4}\, I_0\left(K(\alpha^2-\beta^2)/4\right) \underset{K\to+\infty}{\longrightarrow} \frac{1}{\sqrt{\alpha^2-\beta^2}}\sqrt{\frac{2}{\pi K}} e^{K(1-\beta^2)/2} \tag{3.19}
$$

This means that $\mathcal{P}_{(\alpha,\beta)}(I)$ is no longer the Fourier-Bessel transform of a square integrable function of K; it has switched abruptly from being a classical probability density for intensity to being a distribution, essentially as singular as is permitted by the general considerations of the previous Section!

There is thus no regime in which $\mathcal{P}_{(\alpha,\beta)}(I)$ remains "classical" while ϕ is not - the weakly nonclassical possibility is not realised at all in the family of Gaussian-Wigner states. Even though $\mathcal{P}_{(\alpha,\beta)}(I)$ is a distribution in the squeezed regime, we can obtain the photon number probabilities by analytic continuation starting from the result (3.17) in the nonsqueezed case. The justification is the following. At the level of Wigner distributions we know that the probability $p_{(\alpha,\beta)}(n)$ is the phase space integral of the product of $W_{(\alpha,\beta)}(q,p)$ and the Wigner function $W^{(n)}(q,p)$ for the nth state of the harmonic oscillator [17]:

$$
\begin{aligned}
\hat{\rho} = |n\rangle\langle n| \Rightarrow W^{(n)}(q,p) &= \frac{(-1)^n}{\pi} e^{-(q^2+p^2)} L_n\left(2(q^2+p^2)\right);\\
p_{(\alpha,\beta)}(n) &= 2\pi \int\int dq dp W_{(\alpha,\beta)}(q,p) W^{(n)}(q,p).
\end{aligned}
\tag{3.20}
$$

Here $L_n(.)$ is the nth order Laguerre polynomial. Using the rotational invariance of $W^{(n)}(q,p)$ and eq. (3.7b) for the angular average of $W_{(\alpha,\beta)}(q,p)$, we can reduce $p_{(\alpha,\beta)}(n)$ to a single radial phase space integral:

$$
p_{(\alpha,\beta)}(n) = \frac{(-1)^n}{\pi}\frac{2}{\alpha\beta} 2\pi \int_0^\infty dL \exp\left\{-2L - L\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\right\} L_n(4L) I_0\left(L(\frac{1}{\beta^2}-\frac{1}{\alpha^2})\right) \tag{3.21}
$$

This is valid for all α and β subject to the standard restrictions $\alpha,\ \beta > 1 \quad \alpha\beta \geq 1$. Since we have symmetry in α and β, we may assume with no loss of generality that $\alpha \geq \beta$. Then the asymptotic behaviour (3.11) for $I_0(z)$ as $z \to \infty$ shows that for large L the integrand here behaves like

$$L^{n-1/2} \exp\left\{-2L(1+1/\alpha^2)\right\} \tag{3.22}$$

Thus the integral (3.21) is absolutely convergent for all α and β, and is in fact analytic in these variables (in the appropriate regions of the complex planes).

Having established this, we may now go back to the closed expression (3.17) valid in the nonsqueezed case and analytically continue it to $\beta < 1,\ \alpha\beta \geq 1$. Now from eq. (3.18) we see that the argument z of the hypergeometric function exceeds unity, which lies outside the domain of convergence of the power series expansion of $F(\frac{n+1}{2}, \frac{n}{2}+1; 1; z)$. By analytically continuing to $z > 1$, and keeping track of phases generated in switching from $(\beta^2 - 1)$ to $(1 - \beta^2)$ in the prefactors in eq. (3.17), we find that in the squeezed regime we have different expressions for $p_{(\alpha,\beta)}(n)$ for even n and for odd n:

$$p_{(\alpha,\beta)}(n) = \frac{2}{\sqrt{\pi}} \frac{[(\alpha^2-1)(1-\beta^2)]^{n+1/2}}{(\alpha^2\beta^2-1)^{n+1}} \frac{1}{z^{\frac{n+1}{2}}} \begin{cases} \dfrac{\Gamma(m+1/2)}{m!} F\left(m+1/2, m+1/2; 1/2; \frac{1}{z}\right), & n = 2m, \\ \dfrac{2}{\sqrt{z}} \dfrac{\Gamma(m+3/2)}{m!} F\left(m+3/2, m+3/2; 3/2; \frac{1}{z}\right), & n = 2m+1, \end{cases}$$

$$z = \left(\frac{\alpha^2-\beta^2}{\alpha^2\beta^2-1}\right)^2 > 1, \quad \alpha > \frac{1}{\beta}, \quad \beta < 1. \tag{3.23}$$

Once again we have manifestly nonnegative closed form expressions [8]

The actual expressions for the first few probabilities show the general trend. We find after simplification that, as expected both eq. (3.17) and eq. (3.23) give identical functions of α and β:

$$\begin{aligned}
p_{(\alpha,\beta)}(0) &= 2 \times \left\{(\alpha^2+1)(\beta^2+1)\right\}^{-1/2} \\
p_{(\alpha,\beta)}(1) &= 2(\alpha^2\beta^2-1) \times \left\{(\alpha^2+1)(\beta^2+1)\right\}^{-3/2} \\
p_{(\alpha,\beta)}(2) &= \left\{(\alpha^2-\beta^2)^2 + 2(\alpha^2\beta^2-1)^2\right\} \times \left\{(\alpha^2+1)(\beta^2+1)\right\}^{-5/2} \\
p_{(\alpha,\beta)}(3) &= (\alpha^2\beta^2-1)\left\{3(\alpha^2-\beta^2)^2 + 2(\alpha^2\beta^2-1)^2\right\} \times \left\{(\alpha^2+1)(\beta^2+1)\right\}^{-7/2} \\
p_{(\alpha,\beta)}(4) &= \frac{1}{4}\left\{3(\alpha^2-\beta^2)^4 + 24(\alpha^2-\beta^2)^2(\alpha^2\beta^2-1)^2 + 8(\alpha^2\beta^2-1)^4\right\} \times \left\{(\alpha^2+1)(\beta^2+1)\right\}^{-9/2} \\
p_{(\alpha,\beta)}(5) &= \frac{1}{4}(\alpha^2\beta^2-1)\left\{15(\alpha^2-\beta^2)^4 + 40(\alpha^2-\beta^2)^2(\alpha^2\beta^2-1)^2 + 8(\alpha^2\beta^2-1)^4\right\} \times \left\{(\alpha^2+1)(\beta^2+1)\right\}^{-11/2}
\end{aligned} \tag{3.24}$$

The appearance of the "uncertainty principle factor" $(\alpha^2\beta^2 - 1)$ in $p_{(\alpha,\beta)}(n)$ for odd n alone is immediately understandable: when the uncertainty limit is saturated and $\alpha\beta = 1$, the Gaussian Wigner function $W_{(\alpha,1/\alpha)}(q,p)$ describes the squeezed vacuum, for which it is well known that $p_{(\alpha,1/\alpha)}(n)$ vanishes when n is odd [18]. Conversely, even in the nonsqueezed regime, despite the uniform looking expression (3.17), there is a discrimination between the cases of even and odd n which is seen when the hypergeometric function is worked out in detail. In the limit $\alpha = \beta = 1$, we have of course just the vacuum state, and then $p_{(1,1)}(n)$ vanishes for all $n \geq 1$. This case be seen quite explicitly in the expressions displayed in eq. (3.24).

IV. CONCLUDING REMARKS

We have examined the class of Gaussian-Wigner distributions for a single mode radiation field in quantum optics from the point of view of a recently introduced classification of quantum states into three mutually exclusive types - classical, weakly nonclassical and strongly nonclassical. We have found that only the first and third possibilities arise in this case, corresponding respectively to the nonsqueezed and squeezed situations. As shown elsewhere, there is an interesting class of pure states which give physical examples of the weakly nonclassical type. These are superpositions of the number states of the following general type:

$$|\psi\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{i\beta(n)} |n\rangle, \tag{4.1}$$

where α is any complex number and $\beta(n)$ is a nonlinear function of n. Here the photon number probabilities are independent of $\beta(n)$ and follow the Poisson distribution, so $\mathcal{P}_\psi(I)$ is a delta function:

$$\mathcal{P}_\psi(I) = \delta(I - \alpha^\star \alpha) \tag{4.2}$$

However on the basis of Hudson's Theorem [19] it turns out that the Wigner function $W_\psi(q,p)$, which is not Gaussian, must be negative somewhere, so in turn $\phi(z)$ cannot be nonnegative. This shows that the states (4.1) are weakly nonclassical.

Our result that the centered Gaussian-Wigner distributions are never weakly nonclassical has an important physical consequence. In the regime $\alpha > 1, \beta < 1$ which corresponds to *quadrature* squeezing, since $\mathcal{P}(I)$ is not nonnegative the nonclassical nature of the state *must already show up* in properties of the photon number distribution probabilities $p_{(\alpha,\beta)}(n)$, ie., via phase insensitive quantities. The simplest such signal, namely subpoissonian statistics, does not however display the nonclassicality of the state [8]. We find after simple algebra that the Mandel Q-parameter is always nonnegative:

$$\begin{aligned} Q(\alpha,\beta) &= \frac{\langle \hat{a}^{\dagger^2} \hat{a}^2 \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2}{\langle \hat{a}^\dagger \hat{a} \rangle} \\ &= 2\left\{(\alpha^2-1)^2 + (\beta^2-1)^2\right\}/(\alpha^2+\beta^2-2)^2 \geq 0 \end{aligned} \tag{4.3}$$

There are however (infinitely many) other signatures of a nonclassical photon number distribution, some of which are local in that they involve only a few contiguous probabilities $p(n)$. For example we have the result [7]:

$$\begin{aligned} \mathcal{P}(I) \geq 0 \Rightarrow \quad l(n) &= (n+1)p(n-1)p(n+1) - n(p((n))^2 \geq 0, \\ & n = 1,2,3,... \end{aligned} \tag{4.4}$$

Therefore if any $l(n)$ is negative for some given state, that is evidence for the strongly nonclassical nature of that state. For the states $W_{(\alpha,\beta)}(q,p)$, taking $\alpha = 2$, $\frac{1}{2} < \beta < 1$ as an example, we do find explicitly as shown in Figure 1 that $l(2)$, $l(4)$, $l(6)\cdots$ are negative for some range of values of β before turning positive as β increases; while $l(1)$, $l(3)$, $l(5)\cdots$ do not display such nonclassical behaviour.

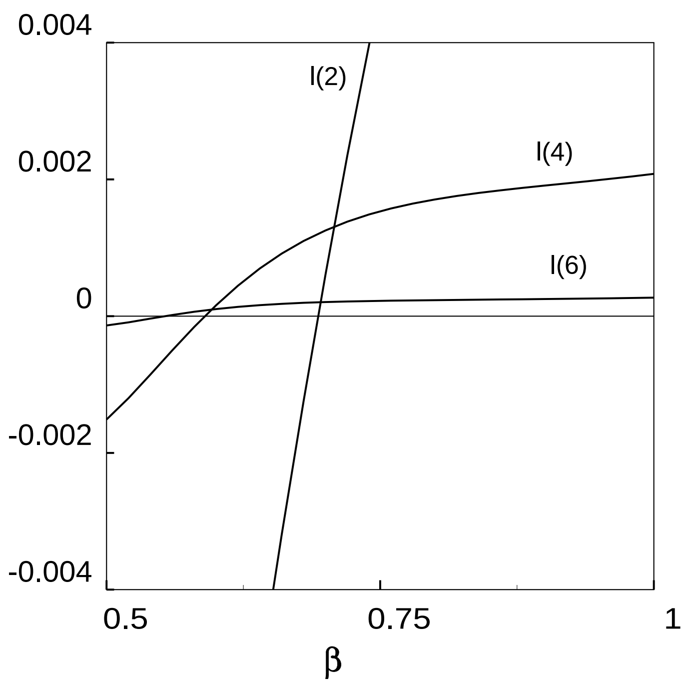

FIG. 1. Violation of the local conditions on photon number distribution in the squeezed regime

It is expected that our conclusions will not be altered drastically if we consider general noncentred Gaussian-Wigner distributions. This aspect and other examples of states and the cases of two or more modes, will be taken up elsewhere.

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